First Quiz+Solution

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ADVANCED ROBOTICS

Chapter II: Rotation of Rigid Bodies

1. (25 points) *Find the axis and the angle of rotation of the proper orthogonal matrix* $\boldsymbol{Q}$ *given below in a certain coordinate frame* $\mathcal{F}$.

$$[\boldsymbol{Q}_{\mathcal{F}}] = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

Answer: According to the Eq. (2.58) of the book, it is apparent that:

$$\mathrm{tr}\,(\mathbf{Q}) = q_{11} + q_{22} + q_{33} = \frac{1}{3}(-1-1-1) = -1,$$

where tr(.) returns the trace of its matrix component. Therefore, according to Eq. (2.69) one has:

$$\cos\phi = \frac{\mathrm{tr}\,(\mathbf{Q})-1}{2} = \frac{-1-1}{2} = -1 \Longrightarrow \sin\phi = 0 \Longrightarrow \phi = (2n-1)\,\pi \quad n \in \mathbb{Z},$$

where ϕ is the angle of rotation.
Moreover, consider the vector $\mathbf{e} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T$ as the unit vector along the axis of rotation, which according to Eq. (2.49) leads to:

$$\mathbf{Q} = -\mathbf{1} + 2\mathbf{e}\mathbf{e}^T \Longrightarrow \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \Longrightarrow$$

$$\frac{1}{3}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix} = 2\begin{bmatrix} e_1^2 & e_1e_2 & e_1e_3 \\ e_2e_1 & e_2^2 & e_2e_3 \\ e_3e_1 & e_3e_2 & e_3^2 \end{bmatrix} \Longrightarrow \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} e_1^2 & e_1e_2 & e_1e_3 \\ e_2e_1 & e_2^2 & e_2e_3 \\ e_3e_1 & e_3e_2 & e_3^2 \end{bmatrix} \Longrightarrow$$

$$e_1^2 = \frac{1}{3} \Longrightarrow e_1 = \pm\frac{\sqrt{3}}{3} \Longrightarrow \mathbf{e} = \pm\frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

upon which the axis of rotation is obtained.

2. (25 points) *The three entries above the diagonal of a 3×3 matrix* $\mathbf{Q}$ *that is supposed to represent a rotation are given below:*

$$q_{12} = \frac{1}{2}, \quad q_{13} = -\frac{2}{3}, \quad q_{23} = \frac{3}{4}$$

Without knowing the other entries, explain why the above entries are unacceptable.

Answer: Obviously, the following condition should be held for the matrix $\mathbf{Q}$ to be orthogonal and then satisfy the requirements of being a rotation matrix:

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}_{3\times 3} \Longrightarrow$$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} q_{11} & \frac{1}{2} & -\frac{2}{3} \\ q_{21} & q_{22} & \frac{3}{4} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ \frac{1}{2} & q_{22} & q_{32} \\ -\frac{2}{3} & \frac{3}{4} & q_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

where the matrix $\mathbf{R}$ has been considered as to checking the orthogonality of the rotation matrix $\mathbf{Q}$. From the above relation, it can be inferred that:

$$r_{11} = 1 \Longrightarrow q_{11}^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{2}{3}\right)^2 = q_{11}^2 + \frac{1}{4} + \frac{4}{9} = q_{11}^2 + \frac{9+16}{36} = q_{11}^2 + \frac{25}{36} = 1 \Longrightarrow q_{11}^2 = \frac{11}{36}$$

$$\Longrightarrow q_{11} = \pm\frac{\sqrt{11}}{6}. \tag{2}$$

Moreover, from Eq. (1) we have:

$$r_{12} = 0 \Longrightarrow q_{11}q_{21} + \frac{1}{2}q_{22} + \left(-\frac{2}{3}\right)\left(\frac{3}{4}\right) = q_{11}q_{21} + \frac{1}{2}q_{22} - \frac{1}{2} = 0. \tag{3}$$

Now substitution of Eq. (2) into Eq. (3) leads to:

$$\pm\frac{\sqrt{11}}{6}q_{21} + \frac{1}{2}q_{22} - \frac{1}{2} = 0 \Longrightarrow \pm\frac{\sqrt{11}}{3}q_{21} + q_{22} - 1 = 0 \Longrightarrow q_{22} = \mp\frac{\sqrt{11}}{3}q_{21} + 1. \tag{4}$$

Furthermore, from Eq. (1) it can be deduced that:

$$r_{22} = 1 \Longrightarrow q_{21}^2 + q_{22}^2 + \left(\frac{3}{4}\right)^2 = q_{21}^2 + q_{22}^2 + \frac{9}{16} = 1. \tag{5}$$

Now substitution of Eq. (4) into Eq. (5) leads to:

$$q_{21}^2 + \left(\mp\frac{\sqrt{11}}{3}q_{21} + 1\right)^2 + \frac{9}{16} = 1 \Longrightarrow q_{21}^2 + \frac{11}{9}q_{21}^2 \mp \frac{2\sqrt{11}}{3}q_{21} + 1 + \frac{9}{16} = 1$$

$$\Longrightarrow \frac{20}{9}q_{21}^2 \mp \frac{2\sqrt{11}}{3}q_{21} + \frac{9}{16} = 0 \Longrightarrow 320q_{21}^2 \mp 96\sqrt{11}q_{21} + 81 = 0 \Longrightarrow$$

$$q_{21} = \frac{\pm 96\sqrt{11} \pm \sqrt{11\,(96)^2 - 4 \times 320 \times 81}}{2 \times 320} = \frac{\pm 96\sqrt{11} \pm \sqrt{11 \times 9216 - 103680}}{640} =$$

$$\frac{\pm 96\sqrt{11} \pm \sqrt{101376 - 103680}}{640} = \frac{\pm 96\sqrt{11} \pm \sqrt{-2304}}{640} = \frac{\pm 96\sqrt{11} \pm 48i}{640} = \frac{3\left(\pm 2\sqrt{11} \pm i\right)}{40},$$

which indicates that for satisfaction of the orthogonality condition, q_{21} must be a complex number, which is not acceptable. Therefore, the matrix $\mathbf{Q}$ cannot be a rotation matrix, thereby completing the proof.

3. (50 points) *The orientation of the end-effector of a given robot is to be inferred from joint-encoder readouts, which report an orientation given by a matrix* $\boldsymbol{Q}$ *in* $\mathcal{F}_1$*-coordinates, namely,*

$$[\boldsymbol{Q}]_1 = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

(a) Show that the above matrix can indeed represent the orientation of a rigid body.

Answer: Two conditions should be checked to be held, for the matrix $\mathbf{Q}$ to be able to represent a rotation. Firstly, the orthogonality condition:

$$\mathbf{Q}\mathbf{Q}^T = \left(\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_{3\times 3},$$

which is satisfied. Moreover, the matrix $\mathbf{Q}$ should be proper, i.e., its determinant should be equal to 1, which is proved as follows:

$$\det(\mathbf{Q}) =$$

$$\frac{1}{27}\{(-1)(-1)(-1) + (2)(2)(2) + (2)(2)(2) - (2)(-1)(2) - (2)(2)(-1) - (-1)(2)(2)\} =$$

$$\frac{1}{27}(-1 + 8 + 8 + 4 + 4 + 4) = \frac{27}{27} = 1.$$

(b) What is $\boldsymbol{Q}$ *in end-effector coordinates, i.e., in a frame* $\mathcal{F}_7$*, if* $\mathcal{Z}_7$ *is chosen parallel to the axis of rotation of* $\boldsymbol{Q}$*?*

Answer: According to the Eq. (2.58) of the book, it is apparent that:

$$\text{tr}(\mathbf{Q}) = q_{11} + q_{22} + q_{33} = \frac{1}{3}(-1 - 1 - 1) = -1,$$

where tr(.) returns the trace of its matrix component. Therefore, according to Eq. (2.69) one has:

$$\cos\phi = \frac{\text{tr}(\mathbf{Q}) - 1}{2} = \frac{-1 - 1}{2} = -1 \Longrightarrow \sin\phi = 0 \Longrightarrow \phi = (2n - 1)\pi \quad n \in \mathbb{Z},$$

where ϕ is the angle of rotation.
Moreover, because $\mathcal{Z}_7$ is chosen parallel to the axis of rotation of $\mathbf{Q}$, the vector $\mathbf{e} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is the unit vector along the axis of rotation, which according to Eq. (2.49) leads to:

$$\mathbf{Q} = -\mathbf{1} + 2\mathbf{e}\mathbf{e}^T = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired rotation matrix.