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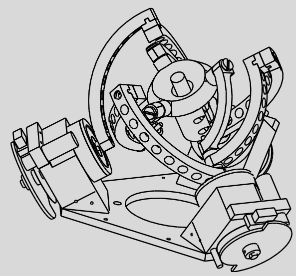

Advanced Robotics
Mathematical Background
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Problem 1

Demonstrate that:

- The eigenvalues of a proper orthogonal matrix $\mathbf{Q}$ lie on the unite circle centered at the origin of the complex plane.
- A proper orthogonal 3 matrix has at least one eigenvalue that is +1.

Problem 2

Consider the following plane in three-dimensional space:

$$x + 2y + 3z = 0 \tag{1.1}$$

Determine:

1. The matrix representing the reflection with respect to this plane;

2. Thee matrix representing the projection into this plane.

Problem 3

Demonstrate the following equations:

1. Equation (2.66). Hint: Expand both sides and show that they result in the same expression.

2. Equations (2.75a) and (2.75 b)

3. Equation (2.78)

Problem 4

Figure 1.1 depicts a tetrahedron in its initial configuration, called configuration I. For our machinery purpose, this should be oriented in such a way that the vertex are as depicted in Fig. 1.1 which leads to configuration II.

1. Since the vertex C is the same for the both configuration, thus the axes of rotation should passes through this point. This axes should also pass through a point, called P, which lies on plane defined by vertex ABD. Determine the position of point P on the aforementioned plane.

2. The corresponding angle of rotation.

3. Find the rotation matrix in a frame for which the axis x, y and z is defined as depicted in configuration I.

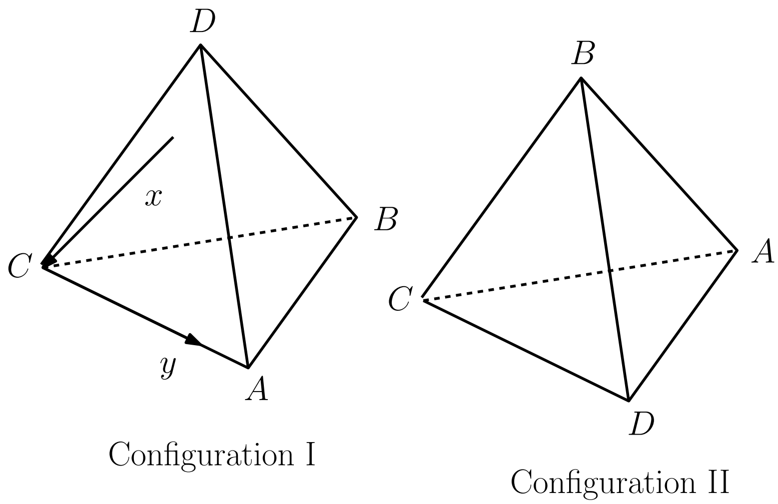

Figure 1.1: Schematic for Problem 4.

Problem 5

A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $O_1(x_1, y_1, z_1)$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $O_2(x_2, y_2, z_2)$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2m above the table top with frame $O_3(x_3, y_3, z_3)$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $O_0(x_0, y_0, z_0)$. Find the homogeneous transformation relating the frame $O_2(x_2, y_2, z_2)$ to the camera frame $O_3(x_3, y_3, z_3)$.

Problem 6

Consider a rigid body which rotate around a fix point with respect to the following matrix:

$$\mathbf{Q} = \begin{bmatrix} c^2 - \frac{1}{3}s^2 & \frac{2}{3}s^2 - \frac{2\sqrt{3}}{3}sc & \frac{2}{3}s^2 + \frac{2\sqrt{3}}{3}sc \\ \frac{2}{3}s^2 + \frac{2\sqrt{3}}{3}sc & c^2 - \frac{1}{3}s^2 & \frac{2}{3}s^2 - \frac{2\sqrt{3}}{3}sc \\ \frac{2}{3}s^2 - \frac{2\sqrt{3}}{3}sc & \frac{2}{3}s^2 + \frac{2\sqrt{3}}{3}sc & c^2 - \frac{1}{3}s^2 \end{bmatrix} \tag{1.2}$$

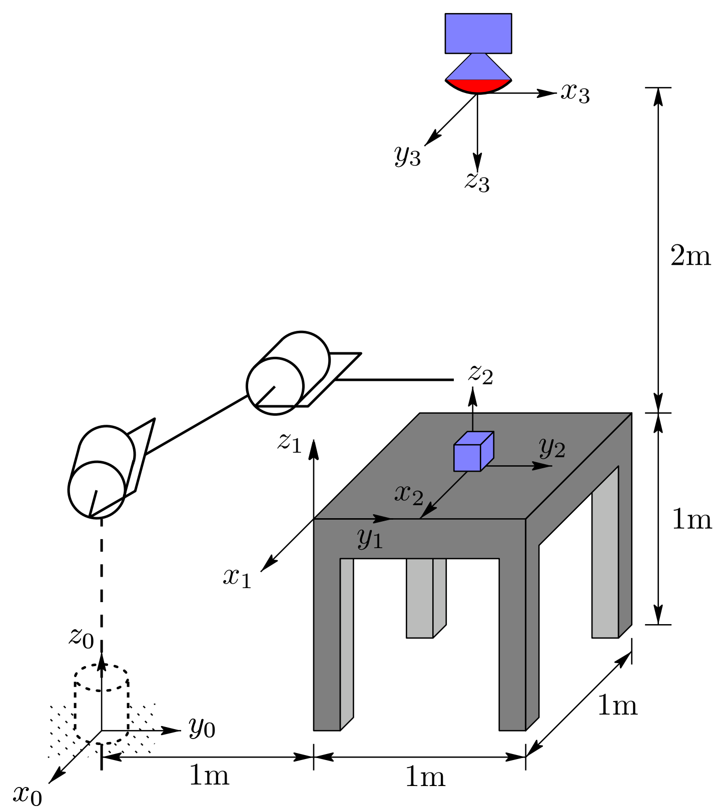

Figure 1.2: Schematic for Problem 5.

where

$$s \equiv \sin\left(\frac{\alpha t}{2}\right) \quad c \equiv \cos\left(\frac{\alpha t}{2}\right) \tag{1.3}$$

where α is a constant value and t represents the time.

1. Find a general expression representing the quadratic invariant with respect to time
2. Deduce an interpretation for the rotation under study
3. Find the quadratic invariant for the rotation at $t_1 = \frac{\pi}{2\alpha}$, $t_2 = \frac{2\pi}{\alpha}$ and $t_1 = \frac{\pi}{\alpha}$.

Problem 7

Given the following 3×3 matrix:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \tag{1.4}$$

1. Show that it is a rotation matrix
2. Determine a unit vector that defines the axis of rotation and the angle (in degrees) of rotation.
3. What are the Euler parameters representing $\mathbf{R}$?

Problem 8

Find the angle and the direction of the axis of rotation of the matrix $\mathbf{Q}$ that takes frame $\mathcal{F}(O, X, Y, Z)$ into frame $\mathcal{F}(C, X_0, Y_0, Z_0)$ of Fig. 2.13. *Hint: Define unit vectors* $\mathbf{i}'$, $\mathbf{j}'$ *and* $\mathbf{k}'$ *parallel to* X', Y' *and* Z'*; then, notice that expressions for* $\mathbf{i}'$ *and* $\mathbf{k}'$ *in terms of* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are straightforward. An expression for* $\mathbf{j}'$ *can be obtained as a cross product of known vectors. As well, convert all your square roots and fractions to decimal form at the outset, using four digits.*

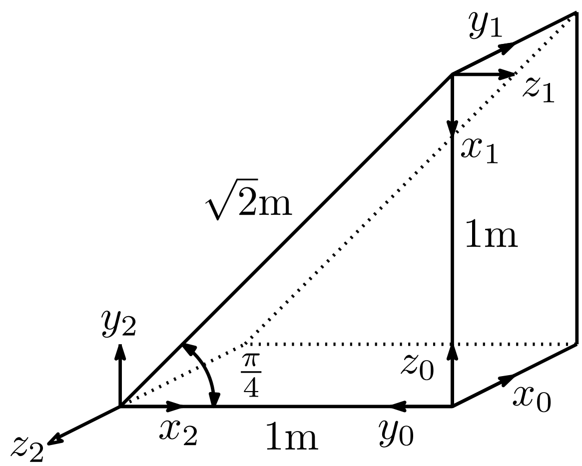

Figure 1.3: Schematic for Problem 12.

Problem 9

Using the quadratic invariant, $\mathbf{a}$ and $\mathbf{a}_0$, or Euler parameters, shows that the rotation matrix can be expressed as follows:

$$\mathbf{Q} = (a_0^2 - \mathbf{a}\mathbf{a}^T)\mathbf{1} + 2\mathbf{a}\mathbf{a}^T + 2a_0\mathbf{1} \times \mathbf{a} \tag{1.5}$$

Problem 10

Do the following exercise from the book: 2.21, 2.27 and 2.30

Problem 11

Find the rotation matrix corresponding to the set of Euler angles and the inverse solution of

1. ZYZ (discuss about the case $\sin\vartheta = 0$).

2. Roll-Pitch-Yaw (discuss about the case $\cos\vartheta = 0$)

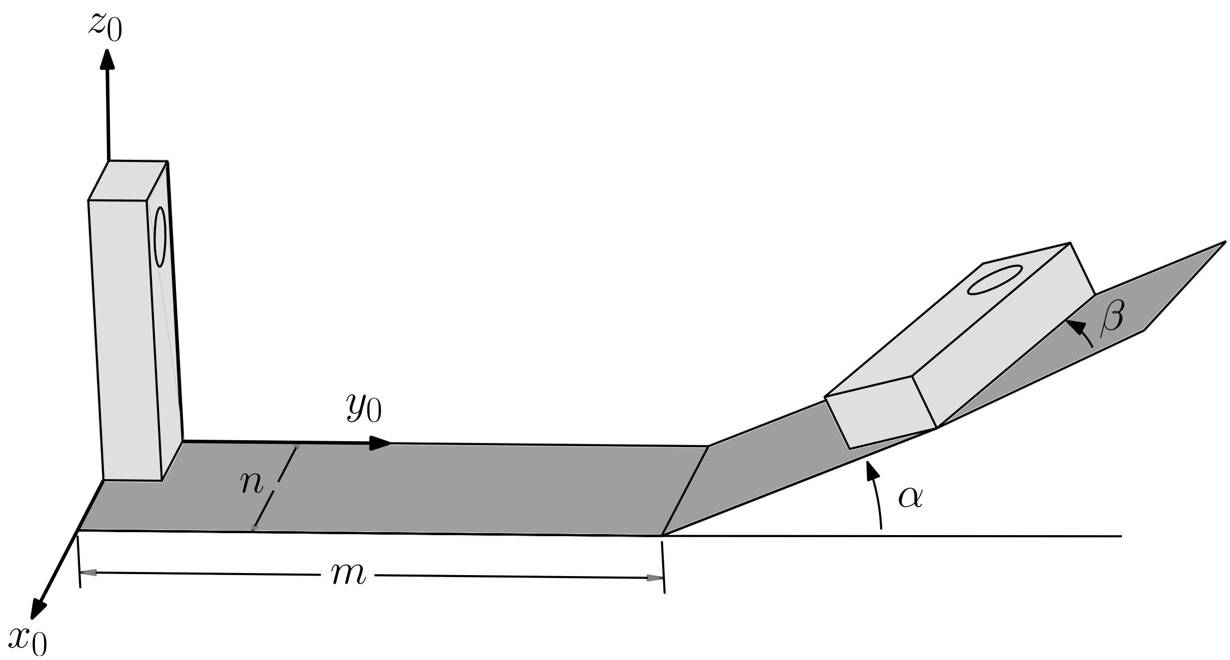

Figure 1.4: Schematic for Problem 15.

Problem 12

Find the homogeneous transformations H_1^0, H_2^0 and H_1^2 representing the transformations among the three frames shown in Fig. 1.3. Find a relation between these transformations.

Problem 13

In Fig. 1.4, a plate is moved from the horizontal base to an inclined surface by a manipulator. With respect to the **i**, **j** and **k** bases, determine:

1. The rotation matrix describing this operation;
2. The axes of rotation and the corresponding rotation angle about this axes.